Spherical polar coordinates

It is convenient to be able to work in spherical polar coordinates as well as in Cartesian (rectangular) coordinates. In order to go from one set to another we need to know expressions for the coordinates of one set in term of those of the other, and also how to express derivatives and volume elements for integration in terms of either set. These results will be presented briefly here.

The polar coordinate r corresponding to a point with Cartesiaon coordinates x, y, z is the distance of that point from the origin. The angle between the z axis and the line from the origin to (x, y, z) is ϑ. When this line is projected onto the x, y plane, the angle between the x axis and the projected line in the x, y plane is φ The expressions for x, y, and z in terms of r, ϑ, and φ result from these definitions and the properties of trigonometric functions:

$$
\begin{aligned}
x &= r \sin\vartheta \cos\varphi \\
y &= r \sin\vartheta \sin\varphi \\
z &= r \cos\vartheta \, .
\end{aligned}
$$

These equations are readily solved to obtain the inverse transformation:

$$
\begin{aligned}
r &= (x^2 + y^2 + z^2)^{1/2} \\
\vartheta &= \arctan \frac{(x^2 + y^2)^{1/2}}{z} \\
&= \arccos \frac{z}{(x^2 + y^2 + z^2)^{1/2}} \\
\varphi &= \arctan \frac{y}{x} \, .
\end{aligned}
$$

We need next to consider how derivatives with respect to different sets of variables are related. Start with a simpler case of two variables. Suppose that we have a function of two variables, $f(x, y)$ and we change to different variables u and v. We can think of either set of variables as being functions of the other set, thus

$$
\begin{aligned}
x &= x(u, v), & y &= y(u, v) \\
u &= u(x, y), & v &= v(x, y) \, .
\end{aligned}
$$

The function $f(x, y)$ is the same as the function g(u,v)

$$
g(u, v) = g[u(x, y), v(x, y)] = f(x, y) \, .
$$

Now suppose that we wish to express derivatives with respect to x and y in terms of derivatives with respect to u and v. The needed relationships are

$$
\begin{aligned}
\frac{\partial f}{\partial u} &= \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial u}\right)_v \\
\frac{\partial f}{\partial v} &= \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial v}\right)_u + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u \, .
\end{aligned}
$$

When the variables being held constant are not indicated, it is assumed that in a derivative with respect to x, y is held constant, etc.

Return now to spherical polar coordinates in three dimensions. If f and g are functions such that

$$f(x,y,z) = g(r,\vartheta,\varphi) = g[r(x,y,z),\vartheta(x,y,z),\varphi(x,y,z)] \ ,$$

then the derivative of f with respect to x can be expressed as

$$\left(\frac{\partial f}{\partial x}\right)_{y,z} = \left(\frac{\partial f}{\partial r}\right)_{\vartheta\varphi}\left(\frac{\partial r}{\partial x}\right)_{y,z} + \left(\frac{\partial f}{\partial \vartheta}\right)_{r\varphi}\left(\frac{\partial \vartheta}{\partial x}\right)_{y,z} + \left(\frac{\partial f}{\partial \varphi}\right)_{r\vartheta}\left(\frac{\partial \varphi}{\partial x}\right)_{y,z} \ .$$

Corresponding expressions can be obtained by interchanges of x, y, and z. From the equations relating the two sets of coordinates we find that

$$\begin{array}{lll}
\frac{\partial x}{\partial r} = \sin\vartheta\cos\varphi & \frac{\partial y}{\partial r} = \sin\vartheta\sin\varphi & \frac{\partial z}{\partial r} = \cos\vartheta \\
\frac{\partial x}{\partial \vartheta} = r\cos\vartheta\cos\varphi & \frac{\partial y}{\partial \vartheta} = r\cos\vartheta\sin\varphi & \frac{\partial z}{\partial \vartheta} = -r\sin\vartheta \\
\frac{\partial x}{\partial \varphi} = -r\sin\vartheta\sin\varphi & \frac{\partial y}{\partial \varphi} = r\sin\vartheta\cos\varphi & \frac{\partial z}{\partial \varphi} = 0
\end{array}$$

and

$$\begin{array}{lll}
\frac{\partial r}{\partial x} = \sin\vartheta\cos\varphi & \frac{\partial r}{\partial y} = \sin\vartheta\sin\varphi & \frac{\partial r}{\partial z} = \cos\vartheta \\
\frac{\partial \vartheta}{\partial x} = \frac{1}{r}\cos\vartheta\cos\varphi & \frac{\partial \vartheta}{\partial y} = \frac{1}{r}\cos\vartheta\sin\varphi & \frac{\partial \vartheta}{\partial z} = -\frac{1}{r}\sin\vartheta \\
\frac{\partial \varphi}{\partial x} = -\frac{\sin\varphi}{r\sin\vartheta} & \frac{\partial \varphi}{\partial y} = \frac{\cos\varphi}{r\sin\vartheta} & \frac{\partial \varphi}{\partial z} = 0 \ .
\end{array}$$

These expressions allow us to express Cartesian derivatives in terms of spherical polar coordinates as

$$\begin{aligned}
\frac{\partial}{\partial x} &= \sin\vartheta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\vartheta\cos\varphi\frac{\partial}{\partial\vartheta} - \frac{1}{r}\frac{\sin\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi} \\
\frac{\partial}{\partial y} &= \sin\vartheta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\vartheta\sin\varphi\frac{\partial}{\partial\vartheta} + \frac{1}{r}\frac{\cos\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi} \\
\frac{\partial}{\partial z} &= \cos\vartheta\frac{\partial}{\partial r} - \frac{1}{r}\sin\vartheta\frac{\partial}{\partial\theta} \ .
\end{aligned}$$

Laplacian

The Laplacian operator $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ can be obtained by squaring and adding the individual derivative *operators*. Note that order must be carefully preserved. For example

$$\frac{\partial^2}{\partial x^2} = \left(\sin\vartheta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\vartheta\cos\varphi\frac{\partial}{\partial\vartheta} - \frac{1}{r}\frac{\sin\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right)^2$$

will include terms like

$$\left[\frac{1}{r}\cos\vartheta\cos\varphi\frac{\partial}{\partial\vartheta}\right]^2 = \frac{1}{r^2}\cos^2\vartheta\cos^2\varphi\frac{\partial^2}{\partial\vartheta^2} + \frac{1}{r^2}\sin\vartheta\cos\vartheta\cos^2\varphi\frac{\partial}{\partial\vartheta} \ .$$

The results are summarized in the following table, where the entries are the coefficients *preceding* the derivatives on the left in the expression for the derivative at the top of the column.

	$\frac{\partial^2}{\partial x^2}$	$\frac{\partial^2}{\partial y^2}$	$\frac{\partial^2}{\partial z^2}$	∇^2
$\frac{\partial^2}{\partial r^2}$	$\sin^2\vartheta\cos^2\varphi$	$\sin^2\vartheta\sin^2\varphi$	$\cos^2\vartheta$	1
$\frac{\partial^2}{\partial\vartheta^2}$	$\frac{\cos^2\vartheta\cos^2\varphi}{r^2}$	$\frac{\cos^2\vartheta\sin^2\varphi}{r^2}$	$\frac{\sin^2\vartheta}{r^2}$	$\frac{1}{r^2}$
$\frac{\partial^2}{\partial\varphi^2}$	$\frac{\sin^2\varphi}{r^2\sin^2\vartheta}$	$\frac{\cos^2\varphi}{r^2\sin^2\vartheta}$	0	$\frac{1}{r^2\sin^2\vartheta}$
$\frac{\partial^2}{\partial r\partial\vartheta}$	$\frac{2\sin\vartheta\cos\vartheta\cos^2\varphi}{r}$	$\frac{2\sin\vartheta\cos\vartheta\sin^2\varphi}{r}$	$-\frac{2\sin\vartheta\cos\vartheta}{r}$	0
$\frac{\partial^2}{\partial r\partial\varphi}$	$-\frac{\sin\varphi\cos\varphi}{r}$	$\frac{\sin\varphi\cos\varphi}{r}$	0	0
$\frac{\partial^2}{\partial\vartheta\partial\varphi}$	$\frac{-2\cos\vartheta\sin\varphi\cos\varphi}{r^2\sin\vartheta}$	$\frac{2\cos\vartheta\sin\varphi\cos\varphi}{r^2\sin\vartheta}$	0	0
$\frac{\partial}{\partial r}$	$\frac{\cos^2\vartheta\cos^2\varphi+\sin^2\varphi}{r}$	$\frac{\cos^2\vartheta\sin^2\varphi+\cos^2\varphi}{r}$	$\frac{\sin^2\vartheta}{r}$	$\frac{2}{r}$
$\frac{\partial}{\partial\vartheta}$	$\frac{\cos\vartheta(-\sin^2\vartheta\cos^2\varphi+\sin^2\varphi)}{r^2\sin\vartheta}$	$\frac{\cos\vartheta(-\sin^2\vartheta\sin^2\varphi+\cos^2\varphi)}{r^2\sin\vartheta}$	$\frac{\sin\vartheta\cos\vartheta}{r^2}$	$\frac{\cos\vartheta}{r^2\sin\vartheta}$
$\frac{\partial}{\partial\varphi}$	$\frac{\sin\varphi\cos\varphi}{r^2\sin^2\vartheta}$	$-\frac{\sin\varphi\cos\varphi}{r^2\sin^2\vartheta}$	0	0

To compare these results with the form

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}$$

note that

$$\begin{aligned}\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} &= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\\ \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta} &= \frac{1}{r^2}\frac{\partial^2}{\partial\vartheta^2} + \frac{1}{r^2}\frac{\cos\vartheta}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\ .\end{aligned}$$

Integrals

We will also wish to express three-dimensional integral in terms of spherical polar coordinates. To do this we must know the range of each variable, to determine the limits of integration, and also how to change the volume element $dxdydz$ to polar coordinates. In most cases of interest we will be integrating over all space so each of the variables x, y, and z ranges from $-\infty$ to ∞. The corresponding ranges for r, ϑ, and φ follow from their definitions and are

$$0 \le r < \infty, \qquad 0 \le \vartheta < \pi, \qquad 0 \le \varphi \le 2\pi\ .$$

Let $g(r,\vartheta,\varphi) = f[x(r,\vartheta,\varphi), y(r,\vartheta,\varphi), z(r,\vartheta,\varphi)]$. It can be shown that when a change of variables is made the volume element is changed by something called the *Jacobian determinant*

$$\int\!\!\int\!\!\int f(x,y,z)\,dxdydz = \int\!\!\int\!\!\int g(r,\vartheta,\varphi)\left|\frac{\partial(x,y,z)}{\partial(r,\vartheta.\varphi)}\right| drd\vartheta d\varphi$$

where the Jacobian determinant is

$$\left|\frac{\partial(x,y,z)}{\partial(r,\vartheta.\varphi)}\right| = \left|\begin{array}{ccc}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\vartheta} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\vartheta} & \frac{\partial y}{\partial\varphi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\vartheta} & \frac{\partial z}{\partial\varphi}\end{array}\right| = r^2\sin\vartheta$$

in this case.

For our purposes this result can simply be accepted as given. We can also consider a geometric argument to show the effect of changing the variables by a small amount. Start with a point determined by x, y, z or r, ϑ, φ and consider changing each variable by a small amount. What volume element will be swept out if each coordinate is changed by a small amount? In the Cartesian case the volume element is simple a rectangular box with sides dx, dy, and dz. If ϑ is changed by an amount $d\vartheta$ while r and φ are constant, the point will sweep out an arc of length $rd\vartheta$. The distance of the point from the z axis is $r\sin\vartheta$, so if φ is now changed by an amount $d\varphi$, the arc length swept out will be $r\sin\vartheta d\varphi$. These two arcs define and area $r^2\sin\vartheta d\vartheta d\varphi$. If r is changed by dr, a thickness dr is added so the volume will be $r^2\sin\vartheta d\vartheta d\varphi dr$.